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Multipliers with Respect to Spectral Measures in Banach Spaces and Approximation. I. Radial Multipliers in Connection with Riesz-Bounded Spectral Measures

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DEDICATED TO PROFESSOR I. J. SCHOENBERG ON THE OCCASION OF HIS 70TH BIRTHDAY, IN FRIENDSHIP AND HIGH ESTEEM

1. INTRODUCTION

In recent years some of the fundamental problems of approximation theory have turned out to be the verification of Jackson-, Bernstein-, Bohr-, and Zamansky-type inequalities for particular approximation processes, the study of the comparison of two different processes with respect to their rate of convergence, as well as the associated problems of nonoptimal and optimal (or saturated) approximation for given processes.

These problems have recently been examined in a unified way in abstract Hilbert spaces (cf. [5]) as well as in the frame of abstract Fourier series in Banach spaces, the processes in question being assumed to possess some multiplier structure (cf. [6, 10–12, 23], see also [3]).

In this paper we would like to continue our previous investigations, this time to extend them to Banach spaces as well as to operators possessing arbitrary (e.g. continuous) spectra (instead of discrete ones). This will be achieved by studying approximation processes to be defined via multipliers with respect to arbitrary spectral measures. Then it is again possible to translate problems such as those mentioned above into the language of uniform multiplier conditions (cf. Section 2). This calls for multiplier criteria, and Section 3 shows that in connection with radial multipliers we may use the criterion already elaborated in our former studies (cf. [6, 23]) and based (cf. [22, 24]) upon the uniform boundedness of Riesz means. Finally, the last two sections are devoted to applications: Whilst Section 4 presents

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examples of concrete processes in arbitrary Banach spaces such as the Riesz means and Bessel potentials, Section 5 specifies some classical examples of concrete Banach space and spectral measures such as those connected with Hankel and multiple Fourier transforms, thus illustrating this unifying approach to the subject. In a forthcoming paper we shall deal with more intricate expansions including variation diminishing kernels, their treatment being based upon their fundamental representation in terms of (two-sided) Laplace transforms, a theory initiated and developed since 1930 by Schoenberg (see e.g. [18]).

2. General Theory

Let X be a Banach space with norm $\|\cdot\|$, and $[X]$ the algebra of all bounded linear operators of X into itself. Consider on X strong approximation processes $\{T_\rho\}_{\rho>0} \subset [X]$; thus, $\lim_{\rho\to\infty} \|T_\rho f - f\| = 0$ for each $f \in X$. To discuss the above topics, it is appropriate to look for some auxiliary Hilbert space H with $H \cap X$ nonempty and dense in H and X, respectively (e.g. in Section 5.2 we shall study the particular situation $H = L^2(\mathbb{R}^n)$ and $X = L^p(\mathbb{R}^n)$, $1 \leq p < \infty$). The given approximation processes $\{T_\rho\}$ are then to be generated by some spectral measure E on H via the following procedure.

Let Σ be the family of Borel sets σ in $\mathbb{R}^n$, the (real) Euclidean n-space (elements of $\mathbb{R}^n$ are usually denoted by $u = (u_1, ..., u_n)$, v, x, elements of $\mathbb{R}^1$ by s, t). Let E be a (countably additive, bounded) spectral measure for H in $\mathbb{R}^n$, i.e., $E(\sigma) \in [H]$ for each $\sigma \in \Sigma$ and ($\varnothing$ being the void set, I the identity mapping)

$$\text{(i)} \quad E(\sigma_1 \cap \sigma_2) = E(\sigma_1)\, E(\sigma_2) \qquad \text{for all} \quad \sigma_1, \sigma_2 \in \Sigma,$$

$$\text{(ii)} \quad E(\varnothing) = 0, \qquad E(\mathbb{R}^n) = I,$$

$$\text{(iii)} \quad E\left(\bigcup_{i=1}^{\infty} \sigma_i\right) = \sum_{i=1}^{\infty} E(\sigma_i) \text{ where } \sigma_i \in \Sigma \text{ and } \sigma_i \cap \sigma_j = \varnothing \text{ for } i \neq j. \tag{2.1}$$

Let $L^\infty(\mathbb{R}^n, E)$ be the set of E-essentially bounded, Borel measurable functions τ on $\mathbb{R}^n$. Then the integral $\int_{\mathbb{R}^n} \tau(u)\, dE(u)$ is well defined in the strong operator topology as an element of $[H]$; in fact, the map $\tau \to \int_{\mathbb{R}^n} \tau(u)\, dE(u)$ is a homomorphism of the algebra $L^\infty(\mathbb{R}^n, E)$ into the algebra $[H]$, thus,

$$\int_{\mathbb{R}^n} \tau_1(u)\, \tau_2(u)\, dE(u) = \left[\int_{\mathbb{R}^n} \tau_1(u)\, dE(u)\right]\left[\int_{\mathbb{R}^n} \tau_2(u)\, dE(u)\right] \tag{2.2}$$

for every $\tau_1, \tau_2 \in L^\infty(\mathbb{R}^n, E)$. For these basic facts compare [8, pp. 900, 1930, 2186, particularly the definition of the algebra $\mathfrak{A}$ p. 1964].

Now $\tau \in L^\infty(\mathbb{R}^n, E)$ is called a multiplier on X (with respect to E, H) if to each $f \in H \cap X$ there exists $f^\tau \in H \cap X$ such that

$$f^\tau = \int_{\mathbb{R}^n} \tau(u)\, dE(u) f, \qquad \|f^\tau\| \leqslant A \|f\|. \tag{2.3}$$

Thus, the operator T of $H \cap X$ into $H \cap X$, defined via $Tf = f^\tau$, has (unique) bounded linear extension to all of X, i.e., $T \in [X]$, and in the following we shall not distinguish between T and its extension to all of X. The set of all multipliers τ on X is denoted by $M = M(X, E, H)$, the corresponding set of multiplier operators T by $[X]_M$. Setting

$$\|\tau\|_M = \sup\{\|f^\tau\|; f \in H \cap X, \|f\| \leqslant 1\}, \tag{2.4}$$

M is a commutative Banach algebra with respect to the natural vector operations and pointwise multiplication, isometrically isomorphic to the subspace $[X]_M \subset [X]$.

Apart from the fact that the approximation processes $\{T_\rho\}$ are of multiplier type, we shall be concerned with certain closed operators. To this end, let $L^\infty_{\mathrm{loc}}(\mathbb{R}^n, E)$ be the set of all Borel measurable functions ψ on $\mathbb{R}^n$ which are E-essentially bounded on any compact subset of $\mathbb{R}^n$. Let $\psi \in L^\infty_{\mathrm{loc}}(\mathbb{R}^n, E)$ be arbitrary, and let H^ψ be the set of all $f \in H$ for which there exists $f^\psi \in H$ such that

$$E(\sigma) f^\psi = \int_\sigma \psi(u)\, dE(u) f \tag{2.5}$$

for each bounded $\sigma \in \Sigma$. Obviously, if B^ψ is the operator with domain H^ψ and range in H, defined by $B^\psi f = f^\psi$, then B^ψ is a closed linear operator (with respect to the topology of H).

Now let $\psi \in L^\infty_{\mathrm{loc}}(\mathbb{R}^n, E)$ and let X_0^ψ be the set of all $f \in H^\psi \cap X$ for which $f^\psi \in H \cap X$. Then ψ is said to generate a X-closed B^ψ if the operator $B^\psi: X_0^\psi \to H \cap X$ is closable in the X-topology, thus has closed linear extension, also denoted by B^ψ, with graph $\overline{X_0^\psi \times B^\psi(X_0^\psi)}$, closure taken with respect to $X \times X$. If the domain of (this extension of) B^ψ is denoted by X^ψ, then X^ψ is a Banach subspace of X with respect to the norm $\|f\|_\psi = \|f\| + |f|_\psi$ with seminorm $|f|_\psi$ given by $\|B^\psi f\|$.

An immediate consequence of the fact that all operators in question should be of multiplier-type in the above sense is that all the problems posed at the beginning may be transposed into corresponding ones upon the coefficients τ, ψ in the form of uniform multiplier conditions.

THEOREM 1. *Let $\{T_\rho\} \subset [X]_M$ be associated with multipliers $\{\tau_\rho\} \subset M$.*

(a) *Let* $\{S_\rho\} \subset [X]_M$ *be a further process associated with* $\{\zeta_\rho\} \subset M$. *If there exists a family* $\{\lambda_\rho\} \subset M$ *with* $\|\lambda_\rho\|_M \leqslant A$ *uniformly for* $\rho > 0$ *such that for (almost all)* $u \in \mathbb{R}^n$

$$\tau_\rho(u) - 1 = \lambda_\rho(u)[\zeta_\rho(u) - 1], \tag{2.6}$$

then one has the (global) comparison theorem *(with respect to the rate of convergence)*

$$\|T_\rho f - f\| \leqslant A \,\|S_\rho f - f\| \qquad (f \in X). \tag{2.7}$$

(b) *Let* $\phi(\rho)$ *be a positive, monotonely increasing function on* $(0, \infty)$ *with* $\lim_{\rho\to\infty} \phi(\rho) = \infty$, *and let* $\psi \in L^\infty_{loc}(\mathbb{R}^n, E)$ *generate a X-closed operator* B^ψ. *Then*

$$\phi(\rho)[\tau_\rho(u) - 1] = \psi(u)\,\lambda_\rho(u), \qquad \sup_{\rho>0} \|\lambda_\rho\|_M \leqslant A \tag{2.8}$$

implies the Jackson-type inequality

$$\phi(\rho)\,\|T_\rho f - f\| \leqslant A\,\|f\|_\psi \qquad (f \in X^\psi), \tag{2.9}$$

whereas the condition

$$\psi(u)\,\tau_\rho(u) = \phi(\rho)\,\lambda_\rho(u), \qquad \sup_{\rho>0} \|\lambda_\rho\|_M \leqslant A \tag{2.10}$$

implies the Bernstein-type inequality

$$\|T_\rho f\|_\psi \leqslant A\phi(\rho)\,\|f\| \qquad (f \in X) \tag{2.11}$$

and the condition

$$\psi(u)\,\tau_\rho(u) = \phi(\rho)\,\lambda_\rho(u)[\tau_\rho(u) - 1], \qquad \sup_{\rho>0} \|\lambda_\rho\|_M \leqslant A \tag{2.12}$$

the Zamansky-type inequality

$$\|T_\rho f\|_\psi \leqslant A\phi(\rho)\,\|T_\rho f - f\| \qquad (f \in X). \tag{2.13}$$

Proof. (a) Let $f \in H \cap X$ be arbitrary. Then (with respect to H, cf. (2.2))

$$\begin{aligned} T_\rho f - f &= \int_{\mathbb{R}^n} [\tau_\rho(u) - 1]\, dE(u) f \\ &= \int_{\mathbb{R}^n} \lambda_\rho(u)[\zeta_\rho(u) - 1]\, dE(u) f = L_\rho[S_\rho f - f], \end{aligned}$$

where $L_\rho \in [X]_M$ is associated with $\lambda_\rho \in M$. By the very definition of a multiplier operator, the latter relation holds for all $f \in X$ proving (cf. (2.4)) the assertion.

(b) Since $L_\rho(H \cap X) \subset H \cap X$, it follows that $L_\rho B^\psi f \in H \cap X$ is well defined for each $f \in X_0^\psi$. Hence, by (2.8) one has for any bounded $\sigma \in \Sigma$

$$E(\sigma)[L_\rho B^\psi f] = \int_\sigma \lambda_\rho(u)\, \psi(u)\, dE(u) f$$

$$= \int_\sigma \phi(\rho)[\tau_\rho(u) - 1]\, dE(u) f = E(\sigma)[\phi(\rho)(T_\rho f - f)]$$

so that for each $f \in X_0^\psi$

$$\phi(\rho)[T_\rho f - f] = L_\rho B^\psi f.$$

Let $f \in X^\psi$ be arbitrary. Since by the definition of X^ψ there exists $\{f_i\} \subset X_0^\psi$ such that $\|f_i - f\| \to 0$, $\| B^\psi f_i - B^\psi f \| \to 0$, assertion (2.9) follows. Observing that the hypotheses (2.10) and (2.12) imply $T_\rho(H \cap X) \subset X_0^\psi$ and, thus, $T_\rho(X) \subset X^\psi$ for each $\rho > 0$, assertions (2.11) and (2.13) follow analogously.

Let us mention that if an estimate of type (2.7) is valid, then the process $\{T_\rho\}$ is called better than $\{S_\rho\}$ on X. If also a converse assertion is valid, then the processes are said to be equivalent on X, in notation $T_\rho f - f \approx S_\rho f - f$, $f \in X$.

One may discuss Bohr-type inequalities (cf. [10; 23, p. 53] for treatment in the discrete case) as well as the saturation problem along the very same lines. However, the latter problem will be dealt with more extensively in [7] where we commence with self-adjoint spectral measures, use dual methods, and assume $H \cap X$ to be weakly* dense in X so as to cover also the space $X = L^\infty(\mathbb{R}^n)$ in the Fourier integral case, for example.

3. A Multiplier Criterion

On account of the reformulation of the approximation–theoretical problems in terms of multipliers, the actual problem in the present setting is to derive a suitable multiplier theory. In case of Hilbert spaces, the multipliers are precisely the E-bounded functions. In case X is not a Hilbert space, we follow up the approach of [6, 23, 24] assuming E to be a Riesz bounded spectral measure for X (for use of Riesz means in spectral calculus see also [14a, 17a]). This will lead to criteria concerning (families of) radial multipliers.

Let X, H, E be as in Section 2. Then E is called a (R, α)-bounded spectral measure for X if the operator ($\alpha \geq 0$)

$$(R, \alpha)_\rho = \int_{\mathbb{R}^n} r_\alpha(|u|/\rho)\, dE(u), \qquad r_\alpha(t) = \begin{cases} (1-t)^\alpha, & 0 \leq t \leq 1, \\ 0, & t \geq 1, \end{cases} \tag{3.1}$$

maps $H \cap X$ into $H \cap X$ for each $\rho > 0$, and if there exists a constant C_α such that for all $\rho > 0$ and $f \in H \cap X$

$$\|(R, \alpha)_\rho f\| \leq C_\alpha \|f\|, \tag{3.2}$$

the family $\{(R, \alpha)_\rho\}_{\rho>0}$ being X-measurable in ρ.

To study multipliers in connection with (R, α)-bounded spectral measures, let us introduce BV_1 as the usual class of functions of bounded variation on $[0, \infty]$ and for $\alpha > 0$

$$BV_{\alpha+1} = \Big\{\lambda \in C[0, \infty];\ \lambda^{(\beta)},..., \lambda^{(\alpha-1)} \in AC_{\text{loc}}(0, \infty),\ \lambda^{(\alpha)} \in BV_{\text{loc}}(0, \infty),$$

$$\|\lambda\|_{BV_{\alpha+1}} = \frac{1}{\Gamma(\alpha+1)} \int_0^\infty t^\alpha\, |d\lambda^{(\alpha)}(t)| + \lim_{t\to\infty} |\lambda(t)| < \infty\Big\}, \tag{3.3}$$

using the following notations: $\beta = \alpha - [\alpha]$, $[\alpha]$ being the largest integer less than or equal to α; $C[0, \infty]$ the set of all bounded, uniformly continuous functions for which $\lim_{t\to\infty} \lambda(t) = \lambda(\infty)$ exists; $AC_{\text{loc}}(0, \infty)$ or $BV_{\text{loc}}(0, \infty)$ the set of all functions which are locally absolutely continuous or of bounded variation, respectively. Thus, for integers α, the above definition of $BV_{\alpha+1}$ is immediately clear. In the strictly fractional case, consider the fractional integral of order $1 - \beta$, $0 < \beta < 1$,

$$I_\omega^{1-\beta}[\lambda](s) = \frac{1}{\Gamma(1-\beta)} \int_s^\omega (t-s)^{-\beta} \lambda(t)\, dt,$$

and define the fractional derivative of order β by

$$\lambda^{(\beta)}(s) = \lim_{\omega\to\infty} [-(d/ds)\, I_\omega^{(1-\beta)}[\lambda](s)].$$

Usual successive differentiation of $\lambda^{(\beta)}$ yields strictly fractional derivatives of order $\alpha = [\alpha] + \beta$, i.e.,

$$\lambda^{(\alpha)}(s) = (d/ds)^{[\alpha]}\, \lambda^{(\beta)}(s). \tag{3.4}$$

Thus, the classes $BV_{\alpha+1}$ are well defined for all $\alpha \geq 0$. Note that $BV_{\alpha+1} \subset BV_{\beta+1}$ in the sense of continuous embedding for all β, $0 \leq \beta \leq \alpha$, and that for any $\lambda \in BV_{\alpha+1}$, $\alpha \geq 0$,

$$\lambda(s) = \frac{(-1)^{[\alpha]+1}}{\Gamma(\alpha+1)} \int_s^\infty (t-s)^\alpha\, d\lambda^{(\alpha)}(t) + \lambda(\infty) \tag{3.5}$$

(cf. [23] and the literature cited therein).

THEOREM 2. *Let E be a (R, α)-bounded spectral measure for X for some $\alpha \geqslant 0$, and let $\lambda \in BV_{\alpha+1}$. Then $\lambda(|u|) \in M$, in fact,*

$$\|\lambda(|u|)\|_M \leqslant C_\alpha \|\lambda\|_{BV_{\alpha+1}}. \tag{3.6}$$

Proof. Obviously, $\lambda(|u|) \in L^\infty(\mathbb{R}^n, E)$. According to definition (2.3), let $f \in H \cap X$ be arbitrary and set

$$f^\lambda = \frac{(-1)^{[\alpha]+1}}{\Gamma(\alpha+1)} \int_0^\infty (R, \alpha)_\rho f \cdot \rho^\alpha \, d\lambda^{(\alpha)}(\rho) + \lambda(\infty) f. \tag{3.7}$$

Then f^λ is well defined as an element of $H \cap X$. Indeed, $f^\lambda \in H$ being obvious,

$$\|f^\lambda\| \leqslant [C_\alpha/\Gamma(\alpha+1) \int_0^\infty \rho^\alpha \, |d\lambda^{(\alpha)}(\rho)| + |\lambda(\infty)|] \, \|f\| \leqslant C_\alpha \|\lambda\|_{BV_{\alpha+1}} \|f\|, \tag{3.8}$$

yielding $f^\lambda \in X$. Moreover, in view of (3.5) one obtains

$$\begin{aligned} f^\lambda &= \frac{(-1)^{[\alpha]+1}}{\Gamma(\alpha+1)} \int_0^\infty \left(\int_{|u|\leqslant\rho} \left(1 - \frac{|u|}{\rho}\right)^\alpha dE(u) f \right) \rho^\alpha \, d\lambda^{(\alpha)}(\rho) + \lambda(\infty) f \\ &= \int_{\mathbb{R}^n} dE(u) f \left[\frac{(-1)^{[\alpha]+1}}{\Gamma(\alpha+1)} \int_{|u|}^\infty (\rho - |u|)^\alpha \, d\lambda^{(\alpha)}(\rho) + \lambda(\infty) \right] \\ &= \int_{\mathbb{R}^n} \lambda(|u|) \, dE(u) f. \end{aligned}$$

Extending the operator L of $H \cap X$ into $H \cap X$, defined via $Lf = f^\lambda$, to all of X, the estimate (3.8) states that $\|L\|_{[X]} \leqslant C_\alpha \|\lambda\|_{BV_{\alpha+1}}$. This proves the theorem.

To estimate $BV_{\alpha+1}$-norms in the strictly fractional case seems to be a difficult task. Here a modification of a theorem of Weyl (1917) provides us with a sufficient, easy-to-verify Lipschitz condition.

PROPOSITION 1. *Let λ have compact support and be $([\alpha]+1)$-fold differentiable on $(0, \infty)$. If $\int_0^a t^\alpha |\lambda^{([\alpha]+1)}(t)| \, dt$ is finite for some $a > 0$ and*

$$\int_0^\infty t^\alpha |\lambda^{([\alpha]+1)}(s+t) - \lambda^{([\alpha]+1)}(t)| \, dt = \mathcal{O}(s^\delta),$$

where $0 < s < 1/4$ and $\delta > \alpha - [\alpha] > 0$, then $\lambda(|u|) \in M$, provided E is a (R, α)-bounded spectral measure for X.

Since the hypotheses imply $\lambda \in BV_{\alpha+1}$ (see [23, p. 46]), the result follows.

Concerning uniformly bounded multipliers in connection with Theorem 2, the multiplier norms $\|\lambda_\rho(|u|)\|_M$ are certainly uniformly bounded if

$\|\lambda_\rho\|_{BV_{\alpha+1}} = \mathcal{O}(1)$, $\rho \to \infty$ (cf. Corollary 4.2). Thus, if the multipliers are of Fejér's type, i.e., $\lambda_\rho(|u|) = \lambda(|u|/\rho)$, then $\|\lambda_\rho(|u|)\|_M \leq C_\alpha \|\lambda\|_{BV_{\alpha+1}}$ since the $BV_{\alpha+1}$-norm is invariant with respect to dilations. However, the condition that the multipliers should be of Fejér's type may be weakened considerably.

PROPOSITION 2. *Let E be a (R, α)-bounded spectral measure for X for some $\alpha \geq 0$, and let $\lambda \in BV_{\alpha+1}$. Let $\phi(\rho)$ be positive for $\rho > 0$, and let $\Phi(t)$ be continuous, strictly increasing on $[0, \infty)$ with $\lim_{t\to 0+} \Phi(t) = 0$ and $\lim_{t\to\infty} \Phi(t) = \infty$. Let Φ possess $([\alpha] + 2)$ derivatives on $(0, \infty)$ with*

$$t^k \, |\Phi^{(k+1)}(t)| \leq C\Phi'(t) \qquad (0 \leq k \leq [\alpha] + 1),$$

C being independent of t, and let Φ' be monotone for all $t \geq t_0 \geq 0$. Then $\lambda(\Phi(|u|)/\phi(\rho)) \in M$ uniformly for $\rho > 0$.

The proof follows along the same lines as for Theorem 2 since the functions $(1 - \Phi(t)/\Phi(\rho))^\alpha$ for $0 \leq t \leq \rho$, 0 for $t \geq \rho$ belong to $BV_{\alpha+1}$ uniformly for $\rho > 0$ (see [23, pp. 28, 47]). The result itself is an appropriate modification of Hardy's "second theorem of consistency." Therefore, we say a family to be of Hardy's type if $\lambda_\rho(u) = \lambda(\Phi(|u|)/\phi(\rho))$ with λ, Φ, ϕ satisfying the above hypotheses. Note that $\Phi(t) = t^\gamma, \gamma > 0$, and $\Phi(t) = \log(1 + t)$ are admissible choices but not $\Phi(t) = e^t$.

As is shown by the preceding results, the spaces $BV_{\alpha+1}$ are connected with radial multipliers. In view of Section 2 the problem naturally arises as to derive multiplier criteria for more general functions $\tau \in L^\infty(\mathbb{R}^n, E)$, not only for those possessing a radial structure. However, in this paper we restrict discussion to the latter case.

4. PARTICULAR APPROXIMATION PROCESSES

Let H be a Hilbert space and X be a Banach space such that $H \cap X$ is dense in H and X. Let E be a spectral measure with respect to H which is (R, α)-bounded with respect to X for some $\alpha \geq 0$. In this section we would like to discuss certain particular choices of families $\{T_\rho\}$ and of functions ψ in connection with Theorem 1. In the following, $\mathbb{Z}$, $\mathbb{P}$, $\mathbb{N}$ denote the sets of all, of all nonnegative, of all positive integers, respectively.

4.1. *Comparison Theorems*

Let us revisit some examples already discussed in the particular (Hilbert space-)setting of [5]. The Gauss–Weierstrass operator is defined by

$$W_{2,\rho} f = \int_{\mathbb{R}^n} e^{-(|u|/\rho)^2} \, dE(u) f \qquad (f \in H \cap X), \tag{4.1}$$

which is to be compared with

$$B_\rho f = \int_{\mathbb{R}^n} \left(1 + \frac{|u|}{\rho}\right) e^{-|u|/\rho}\, dE(u)f \qquad (f \in H \cap X). \tag{4.2}$$

Obviously, both are radial and of Fejér's type, and since $\exp\{-t^2\}$, $(1+t)\,e^{-t} \in BV_{j+1}$ for each $j \in \mathbb{P}$ (and, thus, elements of $BV_{\alpha+1}$ for each $\alpha \geqslant 0$), Theorem 2 delivers $\{W_{2,\rho}\}, \{B_\rho\} \subset [X]_M$ uniformly for $\rho > 0$. In order to obtain an estimate of type (2.7), in view of condition (2.6) and Theorem 2 we have to check that $\lambda(t)$, $1/\lambda(t) \in BV_{\alpha+1}$ with

$$\lambda_\rho(u) = \lambda(|u|/\rho), \qquad \lambda(t) = \frac{1 - e^{-t^2}}{1 - (1+t)\,e^{-t}}. \tag{4.3}$$

To sketch computations in the case of this (easy) instance, let $j \in \mathbb{P}$ be arbitrary (cf. [23, p. 56]). Obviously, λ is arbitrarily often differentiable for $t > 0$, in particular,

$$\lambda'(t) = \frac{2te^{-t^2}}{1-(1+t)\,e^{-t}} - \frac{1 - e^{-t^2}}{[1-(1+t)\,e^{-t}]^2}\, te^{-t} = \frac{Z(t)}{[1-(1+t)\,e^{-t}]^2},$$

and repeated differentiation gives $\lambda^{(j+1)}(t) = \mathcal{O}(t^{j+1}e^{-t} + (2t)^{j+1}\,e^{-t^2})$ for $t \to \infty$; thus, $\int_1^\infty t^j\,|\lambda^{(j+1)}(t)|\,dt < \infty$. To discuss the behavior of λ at the origin, an examination of $Z(t)$ gives (at least) $Z^{(k)}(t) = \mathcal{O}(t^{4-k})$ for $t \to 0+$ and any $k \in \mathbb{P}$. Thus, Leibniz' rule yields

$$\lambda^{(j+1)}(t) = \sum_{k=0}^{j} \binom{j}{k} Z^{(j-k)}(t) \left(\frac{d}{dt}\right)^k [1 - (1+t)\,e^{-t}]^{-2} = \mathcal{O}(t^{-j})$$

for $t \to 0+$, since $(d/dt)^k[1-(1+t)\,e^{-t}]^{-2} = \mathcal{O}(t^{-4-k})$, $0 \leqslant k \leqslant j$. Therefore, $\int_0^1 t^j\,|\lambda^{(j+1)}(t)|\,dt < \infty$ which implies $\lambda \in BV_{j+1}$ for any $j \in \mathbb{P}$, and, thus, $\lambda \in BV_{\alpha+1}$ for any $\alpha \geqslant 0$. Since an analogous argument also delivers $1/\lambda(t) \in BV_{\alpha+1}$, Theorems 1(a) and 2 imply the following corollary.

COROLLARY 4.1. *Let E be a (R, α)-bounded spectral measure for some $\alpha \geqslant 0$. Then the processes* (4.1) *and* (4.2) *are equivalent on X, i.e., there exist constants A_1, A_2 such that for all $f \in X$ (and $\rho > 0$)*

$$A_1 \|W_{2,\rho}f - f\| \leqslant \|B_\rho f - f\| \leqslant A_2 \|W_{2,\rho}f - f\|.$$

In connection with Dirichlet's problem for a strip the following operators, which are neither of Fejér's nor of Hardy's type, are of interest (cf. [2, 4, 5, 25])

$$D_{s\{a\},y}f = \int_{\mathbb{R}^n} k_{s\{a\},y}(|u|)\, dE(u)f, \tag{4.4}$$

$$k_{s\{a\},y}(t) = \frac{\sinh(\eta - y)\,t \overset{+}{(-)} \sinh yt}{\sinh \eta t},$$

where $f \in H \cap X$ and $\rho = y^{-1}$, $y \to 0+$, $0 < y < \eta$ (cf. Section 5.2). These are to be compared with the familiar Abel means ($\rho = y^{-1}$, $y \to 0+$)

$$P_y f = \int_{\mathbb{R}^n} e^{-y|u|}\, dE(u) f \qquad (f \in H \cap X), \tag{4.5}$$

which are of Fejér's type. Again it follows immediately that $\{D_{s,y}\}$, $\{D_{a,y}\}$, $\{P_y\} \subset [X]_M$. To establish an estimate of type (2.7), consider

$$\lambda_{s,y}(t) = \frac{1 - k_{s,y}(t)}{1 - e^{-yt}} = 1 - e^{-(\eta - y)t/2} \nu_{s,y}(t)$$

with $\nu_{s,y}(t) = \cosh(yt/2)/\cosh(\eta t/2)$ as well as

$$\lambda_{a,y}(t) = \frac{1 - e^{-yt}}{1 - k_{a,y}(t)} = 1 - e^{-\eta t/2} \nu_{a,y}(t)$$

with $\nu_{a,y}(t) = \cosh(yt/2)/\cosh((\eta - y)t/2)$. Obviously, it suffices to show, e.g., $\nu_{s,y}(t) \in BV_{j+1}$, for each $j \in \mathbb{P}$ uniformly for $0 < y \leqslant \eta/2$. To this end

$$\nu'_{s,y}(t) = \frac{1}{2} \frac{y \sinh(yt/2) \cosh(\eta t/2) - \eta \cos(yt/2) \sinh(\eta t/2)}{\cosh^2(\eta t/2)}.$$

Further differentiations yield that $\nu_{s,y}^{(j+1)}$ is a finite linear combination of terms of type

$$p(y, \eta) \frac{d_1(yt/2) \cdots d_k(yt/2)\, d_{k+1}(\eta t/2) \cdots d_{j+2}(\eta t/2)}{\cosh^{(j+2)}(\eta t/2)},$$

where $p(y, \eta) \leqslant b\eta^{j+1}$ is a certain polynomial in y, η, the d_k being either cosh or sinh, and k is such that $1 \leqslant k \leqslant j + 1$. In any case, one has for $0 < y \leqslant \eta/2$ that with $a_i = y$ or $= \eta$ and for $2 \leqslant i \leqslant j + 2$

$$\frac{d_1(yt/2)}{\cosh(\eta t/2)} \leqslant e^{-\eta t/4}, \qquad \frac{d_i(a_i t/2)}{\cosh(\eta t/2)} \leqslant 1.$$

Thus, there exists a constant B such that $\nu_{s,y}^{(j+1)}(t) \leqslant Be^{-\eta t/4}$ uniformly for $0 < y \leqslant \eta/2$. Therefore, it follows for each $j \in \mathbb{P}$ that $\lambda_{s,y}(t)$, $\lambda_{a,y}(t) \in BV_{j+1}$ uniformly for $0 < y \leqslant \eta/2$, and indeed for $\eta > 0$. As a consequence, we have the following corollary.

COROLLARY 4.2. *Let E be a (R, α)-bounded spectral measure for some $\alpha \geqslant 0$. Then, for $y \to 0+$, the process $\{D_{s,y}\}$ is better than $\{P_y\}$ which in turn* *is better than* $\{D_{\alpha,y}\}$ *on* X, *i.e., there exist constants* A_1, A_2 *such that for all* $f \in X$ *and* $0 < y \leqslant \eta/2$

$$A_1 \| D_{s,y}f - f \| \leqslant \| P_y f - f \| \leqslant A_2 \| D_{\alpha,y} f - f \|.$$

Concerning converse estimates note that $1/\lambda_{s,y}(t)$ is not bounded at the origin. In order to remove this singularity, one may consider, e.g.

$$\lambda_{\alpha,y}(t) = \frac{1 - e^{-yt}}{1 - k_{s,y}(t) + k_y(t)}, \qquad k_y(t) = 2\frac{\sinh yt}{\sinh \eta t}.$$

Since $\| k_y \|_{BV_{j+1}} \leqslant A_3 y$ and $\| \lambda_{\alpha,y} \|_{BV_{j+1}} \leqslant A_2$ for $0 < y \leqslant \eta/2$, one may complete Corollary 4.2 in the following way.

COROLLARY 4.3. *Under the assumption of Corollary 4.2 there exists a constant* A_3 *such that for* $0 < y \leqslant \eta/2$ *and* $f \in X$

$$\| P_y f - f \| \leqslant A_2 \| D_{s,y} f - f \| + A_2 A_3 \| f \| y,$$
$$A_1 \| D_{\alpha,y} f - f \| \leqslant \| P_y f - f \| + A_1 A_3 \| f \| y.$$

Hence, in these cases one has direct estimates only up to a remainder which, however, is of saturation order (cf. [7]).

To treat some classical summation methods of Hardy's type, let $f \in H \cap X$, $\rho > 0$, and suppose Φ to satisfy the conditions of Proposition 2. We consider the Abel–Cartwright means

$$W_{\Phi,\rho} f = \int_{\mathbb{R}^n} w(\Phi(|u|)/\Phi(\rho))\, dE(u) f, \qquad w(t) = e^{-t}, \tag{4.6}$$

the Bessel potentials of order $\beta > 0$

$$L_{\Phi,\beta,\rho} f = \int_{\mathbb{R}^n} b_\beta(\Phi(|u|)/\Phi(\rho))\, dE(u) f, \qquad b_\beta(t) = (1 + t)^{-\beta}, \tag{4.7}$$

and the Riesz means (cf. (3.1)) of order $\kappa > 0$

$$(R, \kappa)_{\Phi,\rho} f = \int_{\mathbb{R}^n} r_\kappa(\Phi(|u|)/\Phi(\rho))\, dE(u) f. \tag{4.8}$$

The particular choices $\Phi(t) = t^\gamma$, $\gamma > 0$, in (4.6), (4.8) and $\Phi(t) = t^2$ in (4.7) yield the standard versions of these means, respectively, (4.6) in turn reproducing (4.1) and (4.5) for $\gamma = 2, 1$. Since $w, b_\beta \in BV_{\alpha+1}$ for any $\beta > 0$, $\alpha > 0$ and $r_\kappa \in BV_{\alpha+1}$ for any $\kappa \geqslant \alpha$, it follows that $\{W_{\Phi,\rho}\}$, $\{L_{\Phi,\beta,\rho}\}$, $\{(R, \kappa)_{\Phi,\rho}\} \subset [X]_M$ for these values.

COROLLARY 4.4. *Let E be a (R, α)-bounded spectral measure for some $\alpha \geqslant 0$, and let Φ satisfy the conditions of Proposition 2. Then for $\beta > 0$, $\kappa > \alpha$ the processes* (4.6)–(4.8) *are equivalent on X, thus, for $f \in X$, $\rho \to 0$*

$$(R, \kappa)_{\Phi,\rho} f - f \approx W_{\Phi,\rho} f - f \approx L_{\Phi,\beta,\rho} f - f.$$

For the proof we only mention that under the present assumptions the functions (cf. [23, pp. 58, 64])

$$\begin{cases} \dfrac{1 - (1 - t)^{\kappa}}{1 - e^{-t}}, & 0 \leqslant t \leqslant 1, \\ (1 - e^{-t})^{-1}, & t \geqslant 1, \end{cases} \qquad \frac{1 - e^{-t}}{1 - (1 + t)^{-\beta}},$$

as well as their reciprocals belong to $BV_{\alpha+1}$.

In this paper an application of the theory of the previous sections will be restricted to the above processes. However, it is obvious that a number of further applications may be given to those approximation processes generated via radial summation methods of some Riesz-bounded $\int dE(u)$.

4.2. *Jackson- and Bernstein-Type Inequalities*

Concerning applications of Theorem 1(b) to approximation processes such as those mentioned above one may formulate the following corollary.

COROLLARY 4.5. *Let E be a (R, α)-bounded spectral measure for some $\alpha \geqslant 0$, let Φ satisfy the conditions of Proposition 2, and suppose that $\psi(u) = [\Phi(|u|)]^{\gamma}$ generates a X-closed operator B^{ψ} for some $0 < \gamma \leqslant 1$. Then for the processes* (4.6) *and* (4.7) *one has the following Jackson-type inequalities (D being a suitable constant)*

$$\|W_{\Phi,\rho} f - f\| \leqslant D[\Phi(\rho)]^{-\gamma} \|f\|_{\psi}, \tag{4.9}$$

$$\|L_{\Phi,\beta,\rho} f - f\| \leqslant D[\Phi(\rho)]^{-\gamma} \|f\|_{\psi}, \qquad (f \in X^{\psi}). \tag{4.10}$$

Since $0 < \gamma \leqslant 1$, one has $t^{-\gamma}(e^{-t} - 1) \in BV_{j+1}$ for each $j \in \mathbb{P}$, and inequality (4.9) follows immediately by Theorems 1(b) and 2. This together with Corollary 4.4 also implies (4.10).

COROLLARY 4.6. *Under the assumptions of Corollary* 4.5 *one has the following Bernstein-type inequalities*

$$\|B^{\psi} W_{\Phi,\rho} f\| \leqslant D[\Phi(\rho)]^{\gamma} \|f\| \qquad (f \in X) \tag{4.11}$$

for all $\gamma \geqslant 0$ and

$$\|B^{\psi} L_{\Phi,\beta,\rho} f\| \leqslant D[\Phi(\rho)]^{\gamma} \|f\| \qquad (f \in X) \tag{4.12}$$

for $0 \leqslant \gamma \leqslant \beta$, D being a suitable constant.

Since for any $j \in \mathbb{P}$ one has $t^\gamma e^{-t} \in BV_{j+1}$ for any $\gamma \geq 0$ and $t^\gamma(1+t)^{-\beta} \in BV_{j+1}$ for any $0 \leq \gamma \leq \beta$ (cf. [23, pp. 56, 65]), these inequalities follow by Theorems 1(b) and 2.

Of course, the present frame also enables one to derive Bernstein-type inequalities which (partly) cover the classical ones dealing with derivatives of trigonometric polynomials or entire functions of exponential type. Indeed, let $C_{00}^\infty([0, \infty))$ be the space of functions which are arbitrarily often differentiable and have compact support on $[0, \infty)$. For $\gamma \geq 0$ let $\lambda(t) \in C_{00}^\infty([0, \infty))$ be such that $\lambda(t) = t^\gamma$ for $0 \leq t \leq 1$ and $= 0$ for $t \geq 2$. Then obviously $\lambda \in BV_{j+1}$ for any $j \in \mathbb{P}$. Therefore, there exists a constant D such that for Φ satisfying the conditions of Proposition 2 one has

$$\left\| \int_{\mathbb{R}^n} \lambda\left(\frac{\Phi(|u|)}{\Phi(\rho)}\right) dE(u)f \right\| \leq D \left\| \int_{\mathbb{R}^n} dE(u)f \right\| \tag{4.13}$$

for any $f \in H \cap X$.

To give an application let us consider the Riesz means (4.8). Then, for any $\kappa > \alpha$, one has $\{(R, \kappa)_{\Phi,\rho}\} \subset [X]_M$ uniformly for $\rho > 0$. Therefore, for any $f \in H \cap X$ and $\gamma \geq 0$

$$\begin{aligned} &\left\| \int_{|u|\leq\rho} [\Phi(|u|)]^\gamma \left(1 - \frac{\Phi(|u|)}{\Phi(\rho)}\right)^\kappa dE(u)f \right\| \\ &\quad \leq D[\Phi(\rho)]^\gamma \left\| \int_{|u|\leq\rho} \left(1 - \frac{\Phi(|u|)}{\Phi(\rho)}\right)^\kappa dE(u)f \right\| \end{aligned} \tag{4.14}$$

in fact, if $\psi(u) = [\Phi(|u|)]^\gamma$ generates a X-closed operator B^ψ, there exists a constant D_1 such that

$$\|B^\psi(R, \kappa)_{\Phi,\rho}f\| \leq D_1[\Phi(\rho)]^\gamma \|f\| \tag{4.15}$$

for any $f \in X$.

Finally, as an application of Zamansky-type inequalities we mention that ($\psi(u) = [\Phi(|u|)]^\gamma, \gamma \geq 1$)

$$\|B^\psi W_{\Phi,\rho}f\| \leq A[\Phi(\rho)]^\gamma \|W_{\Phi,\rho}f - f\| \qquad (f \in H \cap X), \tag{4.16}$$

since $t^\gamma e^{-t}/(1 - e^{-t}) \in BV_{j+1}$ for each $j \in \mathbb{P}$ if $\gamma \geq 1$.

5. Particular Spectral Measures

In this section, applications of the previous results are given by studying certain concrete instances of spaces H, X, and spectral measures E. Rather than give a complete list of possible applications, our aim is to show how the present approach covers diverse classical results in a unified way as well as how it leads to some new results in concrete situations of Fourier analysis.

5.1. Discrete Spectra

Let $\mathbb{Z}^n$ be the n-fold Cartesian product of $\mathbb{Z}$ with elements $m = (m_1, \ldots, m_n)$. Let H be a Hilbert space and $\{P_m ; m \in \mathbb{Z}^n\} \subset [H]$ a complete system of mutually orthogonal projections of H into H so that each $f \in H$ admits an expansion

$$f = \sum_{m \in \mathbb{Z}^n} P_m f \qquad (f \in H). \tag{5.1}$$

Setting, for any Borel set $\sigma \in \Sigma$,

$$E(\sigma) = \sum_{m \in \sigma} P_m, \tag{5.2}$$

it is obvious that E defines a spectral measure on H. Let X be a Banach space such that $H \cap X$ is dense in H and X. The corresponding approximation processes $\{T_\rho\}$ are then given via expansions of type

$$T_\rho f = \sum_{m \in \mathbb{Z}^n} \tau_\rho(m) P_m f \qquad (f \in H \cap X). \tag{5.3}$$

The notion that E is a (R, α)-bounded spectral measure on X now reads

$$\left\| \sum_{|m| < \rho} \left(1 - \frac{|m|}{\rho}\right)^\alpha P_m f \right\| \leqslant C_\alpha \|f\| \qquad (f \in H \cap X). \tag{5.4}$$

It is essentially this particular situation which was dealt with in [6, 11, 12, 23]. Thus, all the results of these papers are here subsumed so that one may revisit multiple Fourier series, diverse classical expansions such as those concerned with Bessel, Laguerre, and Hermite functions, Jacobi polynomials, spherical harmonics, etc. Since Riesz means are bounded if and only if the Cesàro means of corresponding order are bounded, one may replace condition (5.4) by the corresponding one involving Cesàro means (which was in fact the procedure in our previous studies). Finally, let us mention that the restriction to the discrete case implies several simplifications. For example, the spectral measure can be defined on all of X so that no auxiliary Hilbert space is needed, and the operator B^ψ, then defined by $\psi(m) P_m f = P_m B^\psi f$, $m \in \mathbb{Z}^n$, is obviously a closed operator.

5.2. Fourier Convolution Integrals

Let $H = L^2(\mathbb{R}^n)$ and define the Fourier(-Plancherel) transform $\mathfrak{F}(f)$ of $f \in H$ by ($vu = \sum_{k=1}^n v_k u_k$)

$$\lim_{a \to \infty} \left\| \mathfrak{F}(f)(v) - (2\pi)^{-n} \int_{|u| \leqslant a} f(u)\, e^{-ivu}\, du \right\|_H = 0. \tag{5.5}$$

Let $\mathfrak{F}^{-1}$ be its inverse and $\mathfrak{P}_\sigma$, $\sigma \in \Sigma$, be the multiplication projection

$$\mathfrak{P}_\sigma f(u) = p_\sigma(u) f(u), \qquad p_\sigma(u) = \begin{cases} 1, & u \in \sigma, \\ 0, & u \notin \sigma. \end{cases} \tag{5.6}$$

Setting, for arbitrary $\sigma \in \Sigma$,

$$E(\sigma) = \mathfrak{F}^{-1}\mathfrak{P}_\sigma\mathfrak{F}, \tag{5.7}$$

it is a familiar fact (cf. [8, p. 1989]) that E is a spectral measure on H.

Let $X = L^p(\mathbb{R}^n)$, $1 \leqslant p < \infty$, or $X = C_0(\mathbb{R}^n)$, where these spaces are defined as the set of all functions f which are Lebesgue integrable to the pth power or continuous with $\lim_{|u|\to\infty} f(u) = 0$, endowed with standard norms

$$\left\{\int_{\mathbb{R}^n} |f(u)|^p \, du\right\}^{1/p}, \quad 1 \leqslant p < \infty, \qquad \text{or} \qquad \max_{u\in\mathbb{R}^n} |f(u)|,$$

respectively. Then $H \cap X$ is nonempty and dense in H and X.

On these Banach spaces X we would like to consider operators of Fourier convolution type, e.g.

$$Tf(x) = (2\pi)^{-n} \int_{\mathbb{R}^n} f(x-u)\, d\mu(u) \qquad (f \in X) \tag{5.8}$$

with $\mu \in \mathfrak{M}(\mathbb{R}^n)$, the set of all bounded Borel measures on $\mathbb{R}^n$. As an immediate consequence of the standard convolution and inverse theorems (cf. [8, 1993–8]) one has that operators of type (5.8) may be represented on H in terms of the spectral measure (5.7) via

$$T = \int_{\mathbb{R}^n} \mu^\wedge(u)\, dE(u) \tag{5.9}$$

where $\mu^\wedge(u) = (2\pi)^{-n} \int_{\mathbb{R}^n} e^{-iux}\, d\mu(x)$ is the Fourier (–Stieltjes) transform of $\mu \in \mathfrak{M}(\mathbb{R}^n)$. Thus, operators of type (5.8) are subsumable under the frame of the previous sections.

Concerning an application of Theorem 2, the spectral measure (5.7) is (R, α)-bounded on X for $\alpha > (n-1)\,|\,p^{-1} - 2^{-1}\,|$, $1 \leqslant p \leqslant \infty$, the case $p = \infty$ corresponding to $X = C_0(\mathbb{R}^n)$ (cf. [23] and the literature cited therein; Theorem 2 for $p = 1$ and $[\alpha + 1]$-fold monotone λ see [17]). Now one may reconsider all the results of Section 4 for the present H, X, and E.

In connection with Corollary 4.1 we first of all observe that the processes (4.1) and (4.2) are of type (5.8), in fact

$$W_{2,\rho} f(x) = (\rho/2\pi^{1/2})^n \int_{\mathbb{R}^n} f(x-u)\, e^{-(\rho u/2)^2}\, du, \tag{5.10}$$

$$B_\rho f(x) = \frac{\Gamma((n+1)/2)}{\pi^{(n+1)/2}}\, \rho^n \int_{\mathbb{R}^n} f(x-u)\, \frac{n+1}{(1+(\rho u)^2)^{(n+3)/2}}\, du \tag{5.11}$$

is the explicit representation of the extension of these operators from $H \cap X$ to all of X. As is well known, the integral (5.10) is connected with the solution of the heat equation in $\mathbb{R}^n$ with initial value f (upon setting $\rho = t^{-1}$, $t \to 0+$), whereas (5.11) is connected with "Boussinesq's first problem" in the theory of elasticity (cf. [5, 20, (88.29)]).

COROLLARY 5.1. *Let X be one of the spaces $L^p(\mathbb{R}^n)$, $1 \le p < \infty$, or $C_0(\mathbb{R}^n)$. Then there exist constants A_1, A_2 such that the estimates of Corollary* 4.1 *hold for all $f \in X$ and $\rho > 0$.*

As already mentioned, the integrals (4.4) are related to Dirichlet's problem for the strip $\{(x, y); x \in \mathbb{R}^n, 0 < y < \eta\}$ in connection with symmetric and antisymmetric boundary values, i.e., $D_{s,0}f = D_{s,\eta}f = f$ and $D_{a,0}f = -D_{a,\eta}f$, respectively. They are of type (5.8) (cf. [2, 4, 25]); for example, for $n = 1$ one has the explicit representation in the symmetric case ($\omega = \pi/\eta$)

$$D_{s,y}f(x) = \frac{1}{\pi} \int_{-\infty}^{\infty} f(x-u) \frac{\omega \sin \omega y \cdot \cosh \omega u}{\sinh^2 \omega u + \sin^2 \omega y}\, du.$$

Concerning (4.5), $P_y f$ is the solution of Dirichlet's problem for the upper half-space $\{(x, y); x \in \mathbb{R}^n, y > 0\}$ (case $\eta = \infty$) and given by the Poisson integral

$$P_y f(x) = \frac{\Gamma((n+1)/2)}{\pi^{(n+1)/2}} \int_{\mathbb{R}^n} f(x-u) \frac{y}{(u^2 + y^2)^{(n+1)/2}}\, du.$$

Hence, once again one may reformulate Corollaries 4.2.3 the estimates of which, therefore, hold for each element f of the present X-spaces. Thereby, the importance of these global estimates of the operators $D_{s,y} - I$, $D_{a,y} - I$ in terms of $P_y - I$ is based upon the fact (cf. [4, 5]) that they enable one to transfer all the specific approximation-theoretical results, known for the nicely structured process $\{P_y\}$ (forming a semigroup, being of Fejér's type, etc.), to the more complicated processes $\{D_{s,y}\}$, $\{D_{a,y}\}$.

Obviously one may also reformulate the other results of Section 4 in the present instance. However, we restrict ourselves to an application of (4.13) in connection with entire functions of exponential type. The latter ones are holomorphic on $\mathbb{C}^n$ (the n-fold Cartesian product of the complex plane $\mathbb{C}$) and satisfy for arbitrary $z = (z_1, \ldots, z_n) \in \mathbb{C}^n$

$$|f(z)| \le A e^{a(|z_1| + \cdots + |z_n|)}$$

for some positive constants A and a. If the restriction of f to $\mathbb{R}^n$ is pth power integrable, we write $f \in B_{a,p}$, $1 \le p < \infty$. In case $p = 2$ the Paley–Wiener theorem [19, p. 128] states that

$$f(x) = \int_{|v_j| \le a} \mathfrak{F}[f](v)\, e^{ixv}\, dv \qquad (f \in B_{a,2});$$

thus, in particular $E(\sigma) = 0$ on $B_{a,2}$ if $\sigma \in \Sigma$ and $\sigma \cap \{v \in \mathbb{R}^n; |v|^2 \leqslant na^2\} = \emptyset$. Hence, an application of (4.13) yields that for all $f \in B_{a,p} \cap B_{a,2}$

$$\left\| \int_{\mathbb{R}^n} [\Phi(|u|)]^\gamma \, dE(u) f \right\|_p \leqslant D[\Phi(n^{1/2}a)]^\gamma \|f\|_p \tag{5.12}$$

with some constant D independent of a and f. The extension to all of $B_{a,p}$ may be performed in the following way.

To each function $f \in B_{a,p}$ there exists some sequence $\{f_\epsilon\} \subset B_{a+2\epsilon,p}$, say

$$f_\epsilon(x) = f(x) \prod_{i=1}^n \left(\frac{\sin \epsilon x_i}{\epsilon x_i}\right)^2$$

such that (see e.g. [15])

$$\lim_{\epsilon \to 0+} \|(1 - \Delta)^m (f_\epsilon - f)\|_p = 0 \qquad (1 \leqslant p < \infty) \tag{5.13}$$

for each $m \in \mathbb{P}$, Δ being the standard Laplacian $\sum_{i=1}^n (\partial/\partial x_i)^2$ and $(1 - \Delta)^0$ the identity operator. Then it is not difficult to show that the operator B^ψ, as defined in Section 2, is meaningful on any $B_{a,2}$ and may be extended from $B_{a+2\epsilon,2} \cap B_{a+2\epsilon,p}$ to $B_{a,p}$ if one chooses, e.g. $(\alpha > 0)$

$$\begin{aligned} &\text{(i)} \quad \psi(|u|) = |u|^\alpha, \\ &\text{(ii)} \quad \psi(|u|) = (1 + |u|^2)^{\alpha/2}, \\ &\text{(iii)} \quad \psi(|u|) = \log^\alpha(1 + |u|). \end{aligned} \tag{5.14}$$

To this end we modify a device of Lizorkin [15]: Note that $\lambda(t) = \psi(t)(1 + t^2)^{-m} \in BV_{j+1}$ for each $j \in \mathbb{P}$ provided $m > \alpha$. Hence, $\lambda(|u|)$ is a multiplier on $L^1(\mathbb{R}^n)$ and, thus, has to be the Fourier transform of some $\mu \in \mathfrak{M}(\mathbb{R}^n)$. Since $\mathfrak{F}[f_\epsilon] \in L^2(\mathbb{R}^n)$ has compact support one obtains (cf. (2.5) and (5.7))

$$B^\psi f_\epsilon(x) = \int_{\mathbb{R}^n} \lambda(|u|)(1 + |u|^2)^m \, \mathfrak{F}[f_\epsilon](u) \, e^{ixu} \, du,$$

and on account of the convolution structure

$$B^\psi f_\epsilon(x) = (2\pi)^{-n} \int_{\mathbb{R}^n} (1 - \Delta)^m f_\epsilon(x - y) \, d\mu(y), \tag{5.15}$$

which gives the extension of B^ψ for $\epsilon \to 0+$ to all of $B_{a,p}$ by (5.13).

Thus, (5.12) gives, for example (more precisely, first one has to substitute $\Phi(n^{1/2}a)$ by $\Phi(n^{1/2}(a + 2\epsilon))$ and then to let $\epsilon \to 0+$)

$$\| B^{|u|^\alpha} f \|_p \leqslant D(n^{1/2}a)^\alpha \| f \|_p \qquad (f \in B_{a,p};\ \alpha > 0), \quad (5.16)$$

$$\| B^{(1+|u|^2)^{\alpha/2}} f \|_p \leqslant D(1 + na^2)^{\alpha/2} \| f \|_p \qquad (f \in B_{a,p};\ \alpha > 0), \quad (5.17)$$

$$\| B^{\log(1+|u|)} f \|_p \leqslant D \log(1 + n^{1/2}a) \| f \|_p \qquad (f \in B_{a,p}). \quad (5.18)$$

Let us remark that $B^{|u|^\alpha} f$ may be interpreted as a Riesz derivative of order α and $B^{(1+|u|^2)^{\alpha/2}} f$ as a Bessel derivative of order α (cf., e.g., [15]). In case $\alpha = 2m$ the estimate (5.16) in particular yields

$$\| \Delta^m f \|_p \leqslant D(n^{1/2}a)^{2m} \| f \|_p \qquad (f \in B_{a,p}).$$

(5.17) for all $\alpha > 0$, $n \in \mathbb{N}$ coincides with a result of Nikolskii (cf. [17]). Furthermore, (5.16) extends a result of Lizorkin [15] from $\alpha \geqslant 1$ and $n = 1$ to $\alpha > 0$ and arbitrary integral n. Note, however, that Lizorkin obtains the best possible constant $D = 1$ (also for (5.17) in case $\alpha > 2n - 1$, $\alpha = 2k$) which to obtain by our approach seems hardly possible.

5.3. *Hankel Convolution Integrals*

Choose, for $\nu \geqslant 0$ fixed, the Banach space X as

$$L^{p,\nu}(0, \infty) = \left\{ f;\ \| f \|_{p,\nu} = \left\{ \int_0^\infty | f(t) |^p \, d\mu_\nu(t) \right\}^{1/p} < \infty \right\} \qquad (1 \leqslant p < \infty),$$

where $\mu_\nu(t) = [2^{\nu+1/2}\Gamma(\nu + 3/2)]^{-1} t^{2\nu+1}$. Thus, $H = L^{2,\nu}(0, \infty)$, and $H \cap X$ is dense in X and H. Define the Hankel transform $\mathfrak{H}_\nu(f)$ of $f \in H$ by

$$\lim_{a \to \infty} \left\| \int_0^a s^{-\nu} t^\nu f(t)(st)^{1/2} J_{\nu-1/2}(st)\, dt - \mathfrak{H}_\nu(f)(s) \right\|_H = 0; \qquad (5.19)$$

this limit exists for all $f \in H$, and $\mathfrak{H}_\nu$ is self-reciprocal, i.e., $\mathfrak{H}_\nu[\mathfrak{H}_\nu(f)] = f$ (see [1, p. 226]). Setting, for arbitrary $\sigma \in \Sigma$,

$$E(\sigma) = \mathfrak{H}_\nu \mathfrak{P}_{\sigma \cap [0,\infty)} \mathfrak{H}_\nu\,, \qquad (5.20)$$

$\mathfrak{P}_\sigma$ being given by (5.6), it is known (cf. [5]) that E is a spectral measure on H in $\mathbb{R}^1$.

On the above Banach spaces X we would like to consider operators of Hankel convolution type, e.g.

$$Tf(t) = f \# g(t) = \int_0^\infty f(r)\, g(s)\, D(r, s, t)\, d\mu_\nu(r)\, d\mu_\nu(s), \qquad (5.21)$$

where $g \in L^{1,\nu}$ and

$$D(r, s, t) = [\Gamma(\nu)\, \pi^{1/2}]^{-1}\, 2^{(3\nu-5/2)}\Gamma(\nu + 1/2)^2 (rst)^{-2\nu+1}\, A(r, s, t)^{2\nu-2},$$

$A(r, s, t)$ being the area of a triangle whose sides are r, s, t if there is such a triangle, and zero otherwise. The spaces $L^{p,\nu}$, the convolution concept as well as the above definition of the Hankel transform are taken over from Hirschman [14]. As an immediate consequence of the convolution theorem [14] one has (cf. [5]) that operators of type (5.21) may be represented on H in terms of the spectral measure (5.20) via

$$T = \int_0^\infty g^\wedge(s)\, dE(s), \tag{5.22}$$

where $|g^\wedge(s)| \leqslant \|g\|_{1,\nu}$ (see [14]), $g^\wedge$ being given by

$$g^\wedge(s) = \int_0^\infty s^{-\nu} t^\nu g(t) (st)^{1/2}\, J_{\nu-1/2}(st)\, dt.$$

In order to apply the general theory of Sections 3 and 4 we only have to establish the (R, α)-boundedness of the above spectral measure in the X-topology.

To this end let us first discuss the Bochner–Riesz means of $f \in H \cap X$, namely (cf. (4.8))

$$f \# \mathfrak{H}_\nu \left(r_{2,\alpha}\left(\frac{\cdot}{\rho}\right)\right), \qquad r_{2,\alpha}(s) = \begin{cases} (1 - s^2)^\alpha, & 0 \leqslant s \leqslant 1, \\ 0, & s \geqslant 1. \end{cases}$$

Since $\mathfrak{H}_\nu(\rho^{2\nu+1} g(\rho\,\cdot))(s) = \mathfrak{H}_\nu(g)(s/\rho)$ and $\|\rho^{2\nu+1} g(\rho\,\cdot)\|_{1,\nu} = \|g\|_{1,\nu}$, it suffices to examine $\mathfrak{H}_\nu(r_{2,\alpha})(t)$. Now, by [9, p. 26, (33)] one obtains (C being a constant independent of t)

$$\mathfrak{H}_\nu(r_{2,\alpha})(t) = C t^{-\nu-\alpha-1/2} J_{\nu+\alpha+1/2}(t),$$

$$J_\beta(t) = \sum_{j=0}^\infty (-1)^j \frac{(t/2)^{\beta+2j}}{j!\,\Gamma(j + \beta + 1)}.$$

Since for fixed β there holds $J_\beta(t) = \mathcal{O}(t^{-1/2})$ for $t \to \infty$, one obviously has $\mathfrak{H}_\nu(r_{2,\alpha}) \in L^{1,\nu}$ for $\alpha > \nu$. Using Stein's [21] interpolation theorem this yields for $1 \leqslant p < \infty$

$$\|\rho^{2\nu+1} r_{2,\alpha}(\rho\,\cdot) \# f\|_{p,\nu} = \left\| \int_0^\rho (1 - (s^2/\rho^2))^\alpha\, dE(s) f \right\|_{p,\nu}$$
$$\leqslant A_{p,\nu} \|f\|_{p,\nu} \qquad (\alpha > 2\nu\, |1/p - 1/2|).$$

Replacing ρ^2 by ρ' it is clear by the proof to Theorem 2 that $BV_{\lambda+1}$ is again a multiplier class. Observing that $r_\alpha(s) = (1+s)^{-\alpha} r_{2,\alpha}(s)$, it is easy to verify that $(1+s)^{-\lambda} \in BV_{j+1}$ for each $j \in \mathbb{P}$, and, therefore, $(1+s/\rho)^{-\alpha}$ is a multiplier uniformly in $\rho > 0$.

PROPOSITION 3. *Let $f \in L^{2,\nu} \cap L^{p,\nu}$, $1 < p < \infty$, and $\alpha > (\nu+1)|1-2/p|$. Then*

$$\left\| \int_0^\rho (1 - s/\rho)^\alpha \, dE(s) f \right\|_{p,\nu} \leq C_\alpha \|f\|_{p,\nu}.$$

Hence, all the properties required for the results in Sections 3 and 4 are satisfied. In particular Theorem 2 and Propositions 1 and 2 give multiplier criteria for Hankel transforms. Multiplier criteria of the Marcinkiewicz-type follow from Igari [14b] and in weighted L^p-spaces from Guy [13]. But Guy's notations are only consistent with the ones used here in case $p = 2$; $\mathfrak{H}_\nu(f)$ as given above is called a "modified" Hankel transform by Guy. For $1 < p < \infty$, $p \neq 2$, he treats multiplier operators for the "ordinary" Hankel transform

$$\int_0^\infty f(t) \, J_\nu(st)(st)^{1/2} \, dt,$$

which is closely connected with the Fourier transform on the real line; indeed, some type of transplantation approach is applied.

Let us conclude with a brief application to Jackson- and Zamansky-type inequalities for the Bessel potentials with $\Phi(s) = s^2$ and $\beta = \nu + 1$. Thus, $b_{\nu+1}(s^2/\rho^2) = (1 + s^2/\rho^2)^{-\nu-1}$, and (see [14])

$$\mathfrak{H}_\nu(b_{\nu+1}(s^2/\rho^2))(t) = e^{-\rho t} \rho^{2\nu+1} 2^{\nu+1/2} \Gamma(\nu + 1/2)/\Gamma(2\nu + 1)$$

is the kernel corresponding g in (5.21). To give a concrete interpretation of particular $\psi(u)$ occuring in (2.8) and (2.12), let us consider the differential operator

$$\Box f(t) = f''(t) + (2\nu/t) f'(t), \tag{5.23}$$

which coincides with the standard Laplacian for radial functions on $\mathbb{R}^n$ in case $2\nu = n - 1$. For appropriate smooth $f \in L^{2,\nu} \cap L^{p,\nu}$ this differential operator satisfies the relation (see [14])

$$\Box f = -\int_0^\infty s^2 \, dE(s) f.$$

The explicit representation (5.23) of $\Box$ shows that B^{s^2} is an X-closed operator. Thus, since $[1 - b_{\nu+1}(s)] s^{-1}$ and $s \cdot b_{\nu+1}(s)/(1 - b_{\nu+1}(s))$ both belong to BV_{j+1} for each $j \in \mathbb{P}$, one has the following corollary.

COROLLARY 5.2. *Let X, H, E, and* $\square$ *be as above. Then*

(i) $\| L_{s^2,\nu-1,\rho}f - f \|_{p,\nu} \leqslant C_1\rho^{-2} \| \square f \|_{p,\nu} \qquad (f \in X^{s^2}),$

(ii) $\| \square L_{s^2,\nu+1,\rho}f \|_{p,\nu} \leqslant C_2\rho^2 \| L_{s^2,\nu+1,\rho}f - f \|_{p,\nu} \qquad (f \in X).$

As a final remark let us observe that the discrete case as treated, e.g. in [6], was developed in terms of expansions according to projection operators (cf. (5.1)). This approach could also be interpreted in terms of expansions according to (generalized) bases (see [6, Section 2]). For the transition from the discrete to the continuous, and for the subsumation of both continuous and discrete theories in one framework we have chosen the standard approach via (projection-valued) spectral measures, it sufficing for the range of applications considered by us so far. Perhaps it may be possible to cover the material presented by Masani's [16] very modern theory of H-basic, countably additive, orthogonally scattered measures which may be considered as substitutes for (ordinary) bases.

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